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INVESTIGATION OF THE ELECTROCONVECTIVE FLOW OF A WEAKLY CONDUCTING LIQUID WITH UNIPOLAR INJECTION CONDUCTIVITY BY THE FINITE-ELEMENT METHOD

I. A. Ermolaev and A. I. Zhbanov UDC 537.8:532.5

The electroconvective flow of a weakly conducting liquid dielectric in a plane-parallel electrode system is investigated by the finite-element method of Galerkin in the vicinity of the crisis of equilibrium-stability loss. The threshold value of the strength of a homogeneous electric field (corresponding to the occurrence of stationary two-dimensional spatially periodic flow structures) is determined. The empirical relation for the extremum of the current function in the vicinity of the crisis is obtained. The distribution of the volume density of charges over the interelectrode gap in the ascending and descending convective flows is found.

The electric field of a plane-parallel electrode system can give rise to the vortex spatially periodic self-structures (resembling Benard cells) of electroconvective flow of a weakly conducting liquid dielectric [1, 2]. The structures are stable in dielectric liquids with specific conductivities of 10^{-13} to 10^{-8} $(\Omega\cdot\text{cm})^{-1}$ and occur in a crisis manner as a result of the stability loss by the equilibrium of a plane layer [3]. The threshold of their occurrence depends on the conductivity, the viscosity, and the chemical composition of the liquid and it is 0.5–5 kV/cm in the majority of commercial dielectrics [4]. The structures exist up to breakdown voltages of 40–60 kV/cm. Unipolar injection of charges from the cathode is responsible for electroconvective flow for many dielectric liquids under isothermic conditions [5, 6]. In this case, a like electric charge is formed near the electrode; this charge either has time to relax due to the ohmic current and the interaction with the opposite ions of the liquid or provokes electroconvective motion in the form of spatially periodic structures.

The hydrostatic stability of a weakly conducting liquid with injection conductivity has been investigated by methods of the linear stability theory for a cylindrical layer [7], for a spherical layer [8], and for a plane-parallel system of electrodes with a symmetric space-charge distribution [9]. However, the linear stability theory is based on the assumption of the smallness of the occurring disturbances and their exponential growth with time, which holds only in the vicinity of the crisis. The evolution of finite-amplitude disturbances can be determined just on the basis of solution of complete nonlinear equations or their numerical analogs. Thus, in [10], electroconvective flow in the working section of an electrohydrodynamic pump with point ionizers has been investigated by the control-volume method. In [11, 12], consideration has been given to the stationary electroconvective structures formed in a weakly conducting liquid as a consequence of unipolar injection from the cathode in a cylindrical cavity. The finite-difference method was employed as the numerical method. The branching of supercritical regimes of flow has been investigated in [12].

In the present work, we consider electroconvection in a weakly conducting liquid with unipolar injection conductivity in a plane-parallel electrode system. We employ the finite-element method in Galerkin's form which enables us to efficiently approximate curvilinear boundaries for different types of boundary conditions and to solve problems of bunching of the grid in the regions of large gradients of the functions sought. The two-dimensional nonstationary equations of electroconvection are formulated in the velocity vortex–current function–volume charge density variables, which enables us not to consider the pressure and ensures a more efficient solution in many cases.

Mathematical Model. The general system of equations of passage of the current in isothermal viscous incompressible weakly conducting fluids (liquids) has the form

N. G. Chernyshevskii Saratov State University, Saratov, Russia; email: ZhbanovAI@info.sgu.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 75, No. 5, pp. 96–99, September–October, 2002. Original article submitted February 22, 2002.

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\nabla)\,\mathbf{V}\right)=-\nabla p+\mu\nabla^2\,\mathbf{V}+q\,\mathbf{E}\,, \tag{1}$$

$$\nabla\,\mathbf{V}=0\,, \tag{2}$$

$$\varepsilon\,\nabla\,\mathbf{E}=q\,,\quad \mathbf{E}=-\nabla\varphi\,, \tag{3}$$

$$\frac{\partial q}{\partial t}+\nabla\,(\sigma_0\,\mathbf{E}+q\,\mathbf{V}-D_{\mathrm{ion}}\,\nabla q)=0\,. \tag{4}$$

The system consists of Navier–Stokes equations which involve the density of Coulomb forces, the Gauss law and the condition of potentiality of the electric field, and the law of conservation of electric charge. The processes in question are considered to be quasielectrostatic, i.e., they are related to characteristic times that are much longer than the time of propagation of electromagnetic waves. The electric-field energy is substantially higher than the magnetic-field energy.

Under isothermal external conditions, Joule heating becomes the main mechanism of breaking of isothermicity; however it is very small for the insulating liquids in question, which is noted in many works, for example, in [6, 13] and others. In particular, the temperature changes due to the Joule heating in an electric-wind regime at 10 kV in transformer oil are no higher than 0.005^{o}C [6]. Such low heating ensures the smallness of the buoyancy force and cannot produce an appreciable change in the conductivity of the medium; consequently, it is not a substantial driving force. Furthermore, the density and the permittivity of the medium can be considered to be constant under isothermal conditions, which eliminates electrostrictional forces and forces related to the permittivity gradient. The mobility of charges, the permittivity, and the viscosity depend weakly on the field strength, which enables us to consider these quantities to be constant up to breakdown voltages. The diffusion coefficient of ions was also taken to be negligibly small.

Equations (1)–(4) were solved in a two-dimensional formulation in the velocity vortex–current function–volume charge density variables. Introducing the function $\omega = \partial V_x/\partial y - \partial V_y/\partial x$ and taking into account the continuity equation, we can eliminate the pressure and decrease the number of equations in the system. For the plane-parallel electrode system the equations take the form

$$\frac{\partial\omega}{\partial t}+V_x\frac{\partial\omega}{\partial x}+V_y\frac{\partial\omega}{\partial y}=\nu\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)+\frac{E_x}{\rho}\frac{\partial q}{\partial y}-\frac{E_y}{\rho}\frac{\partial q}{\partial x}\,, \tag{5}$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\omega\,, \tag{6}$$

$$V_x=\frac{\partial\psi}{\partial y}\,,\quad V_y=-\frac{\partial\psi}{\partial x}, \tag{7}$$

$$\frac{\partial q}{\partial t}+V_x\frac{\partial q}{\partial x}+V_y\frac{\partial q}{\partial y}=-\frac{\sigma_0}{\varepsilon}\,q\,. \tag{8}$$

The linear injection of the charges $q = \eta E_y$ was assumed as the boundary conditions on the cathode; the discharging of the ions was considered to be instantaneous on the anode, i.e., $q = 0$.

The "sticking" conditions $\psi = 0$ and $\partial\psi/\partial n = 0$ were provided at solid impermeable boundaries (on the electrodes).

The values of the velocity vortex at the boundary were determined from the Woods formula [14]. The symmetry conditions $\partial q/\partial n = 0$, $\omega = 0$, and $\psi = 0$ were specified at the lateral boundaries of the convective unit cell.

The initial conditions had the form $\omega = 0$, $\psi = 0$, and $q = 0$.

Methods of Solution. System (5)–(8) was solved successively; the volume density of charges, the velocity vortex, and the current function were approximated by a linear combination of time-independent basis functions (shape functions) on linear triangular finite elements. The discrepancy resulting from the approximation is orthogonalized relative to the basis functions, which yields a system of equations of the form

$$\frac{\partial}{\partial t}[C]\{\Phi\} + [K]\{\Phi\} + \{F\} = 0 .$$

The elements of the matrix $[K]$ and of the vector $\{F\}$ have the following form:

for the velocity vortex

$$k_{ij} = \int\limits_{\Omega_e} \left(V_x N_i \frac{\partial}{\partial x} N_j + V_y N_i \frac{\partial}{\partial y} N_j - \nu \frac{\partial}{\partial x} N_i \frac{\partial}{\partial x} N_j - \nu \frac{\partial}{\partial y} N_i \frac{\partial}{\partial y} N_j \right) d\Omega , \quad f_i = \int\limits_{\Omega_e} N_i \left(\frac{E_x}{\rho} \frac{\partial q}{\partial y} - \frac{E_y}{\rho} \frac{\partial q}{\partial x} \right)_e d\Omega ;$$

for the current function

$$k_{ij} = -\int\limits_{\Omega_e} \left(\frac{\partial}{\partial x} N_i \frac{\partial}{\partial x} N_j + \frac{\partial}{\partial y} N_i \frac{\partial}{\partial y} N_j \right) d\Omega , \quad f_i = \int\limits_{\Omega_e} N_i \omega_e d\Omega ;$$

for the volume density of charges

$$k_{ij} = \int\limits_{\Omega_e} \left(\frac{\sigma_0}{\varepsilon} N_i N_j + V_x N_i \frac{\partial}{\partial x} N_j + V_y N_i \frac{\partial}{\partial y} N_j \right) d\Omega , \quad f_i = 0 .$$

The totally implicit scheme was employed for the time approximation.

Results. Accurate determination of the critical (threshold) strength of the electric field which corresponds to the stability loss by equilibrium and to the onset of the formation of spatially periodic structures is an important problem in theoretical and practical terms. Such a problem has been solved experimentally, for example, for the transformer-oil solution of molecular iodine [6]. It can also be solved by the method of numerical experiment. An analogous means of determining the threshold of occurrence of Rayleigh-thermoconvection structures has been employed in [15, 16] and other works. Considering that the root law

$$\psi_{\mathrm{m}} \sim (E - E^*)^{1/2} ,$$

holds in the vicinity of the crisis of equilibrium-stability loss, we can extrapolate this dependence to the zero value of ψ_{m} and obtain the threshold strength E^* with a good accuracy.

The numerical experiment was carried out for the liquid with unipolar injection conductivity for $\sigma_0 = 10^{-12}$ $(\Omega\cdot\mathrm{cm})^{-1}$, where σ_0 is the initial conductivity of the liquid measured by the linear portion of the volt-ampere characteristic. This value of the conductivity ensures the lowest threshold of the flow and the highest breakdown voltage [3]. The parameters of the liquid were as follows: $\nu = 7.05\cdot10^{-1}$ cm^2/sec, $\rho = 8.9\cdot10^{-4}$ kg/cm^3, $\varepsilon = 2.3$, and $\eta = 1.25\cdot10^{-13}$ C/(V·cm^2); this corresponds to the transformer-oil solution of molecular iodine (the liquid-layer thickness was 1 cm).

The stationary structures of electroconvective flow were modeled according to the established method by solving the nonstationary problem (5)–(8) for nearly threshold average strengths of the homogeneous field. We recorded the maximum values of the current function. The calculations were carried out on a 20×40 finite-element grid with a time step of 0.01 sec.

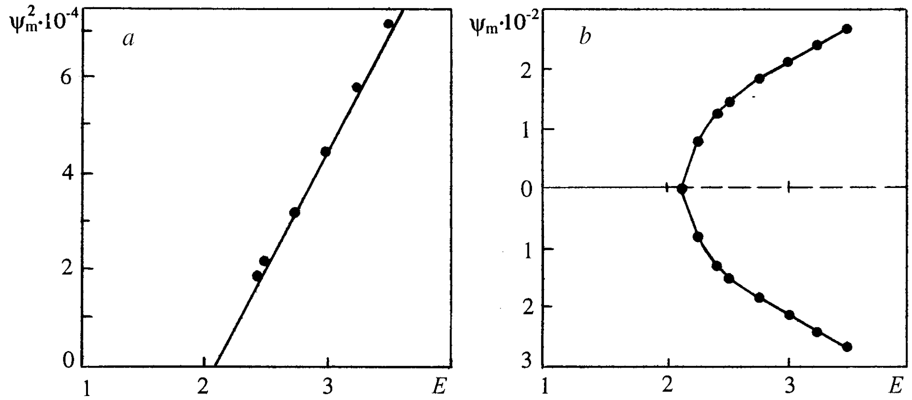
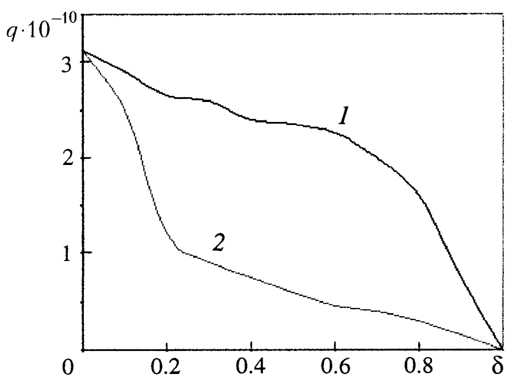


Fig. 1. Square of the maximum current function ψ_m^2 (a) and extremum of the current function ψ_m (b), vs. modulus of the strength of the homogeneous electric field E. ψ_m, $cm^2 \cdot sec$; E, kV/cm.



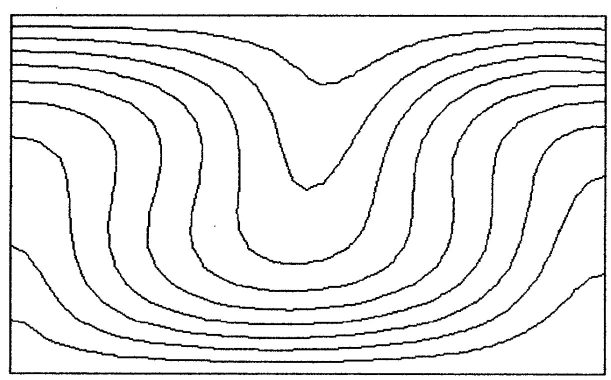


Fig. 2. Distribution of the volume density of charges q over the interelectrode gap δ in the descending (1) and ascending (2) convective flows (δ = 0, cathode, δ = 1, anode). q, C/cm^3; δ, cm.

Fig. 3. Isolines of the field of volume density of charges. At the bottom, cathode, at the top, anode.

The threshold strength of the homogeneous field has been computed for spatially periodic structures with a wavelength of $l = 2$ (wave number $k = \pi$), which corresponds to the convective unit cell with an aspect ratio of $L/\delta = 2$.

We carried out six numerical calculations to determine the threshold strength. The maximum current functions characterizing the intensity of structural circulation have been obtained for average electric-field strengths of 2.40, 2.45, 2.50, 2.75, 3.00, 3.25, and 3.50 kV/cm. Figure 1a shows the linear dependence of the square of the maximum current function on the average field strength, which indicates a "soft" instability. Extrapolation of the dependence to the zero value of ψ_m enables us to obtain a threshold strength of the electric field of $E^* \approx 2.1$ kV/cm, which agrees with experimental results [6]. Approximation was carried out according to the least-squares method. The root law for the maximum current function in the vicinity of the crisis

$$\psi_m \approx 2.408 \cdot 10^{-2} (E - E^*)^{1/2}$$

has been obtained.

Figure 1b gives the bifurcation diagram of the maximum value of the current function versus the electric-field strength. By varying the initial state of the dielectric layer we have obtained the lower branch of the diagram corresponding to the opposite rotation of vortices. The horizontal axis corresponds to the equilibrium solution which becomes unstable for a field strength higher than E^*.

Figure 2 shows changes in the volume density of charges over the interelectrode gap in the descending (1) and ascending (2) convective flows for an average field strength of E = 2.5 kV/cm. The wavelength of the structures is l = 1.3. Figure 3 shows the isolines of the field of the volume density of charges for the indicated flow structures.

NOTATION

ρ, density; **V**, velocity vector; p, pressure; μ, dynamic viscosity; **E** vector of electric-field strength; q, volume density of charges; t, time; ε, permittivity; φ, electric potential; σ_0, initial conductivity; D_{ion}, diffusion coefficient of ions; ω, velocity vortex; V_x and V_y, components of the velocity vector; ν, kinematic viscosity; E_x and E_y, components of the vector of electric-field strength; x, y, Cartesian coordinates; ψ, current function; η, injection coefficient; n, external normal to the boundary; [C], damping matrix; [K], stiffness matrix; {F}, force vector; {Φ}, vector of nodal values; N_i and N_j, shape functions; Ω_e, area of the element e; ω_e, velocity vortex on the element e; ψ_{m}, maximum current function; E^*, critical value of the strength of the homogeneous electric field; l, wavelength of spatially periodic structures; k, wave number of the structures; L, horizontal dimension of the convective cell; δ, layer thickness. Subscripts: ion, ion; m, maximum; e, finite element.

REFERENCES

1. M. J. Cross and J. E. Porter, *Nature*, **212**, No. 5068, 1343–1345 (1966).
2. G. A. Ostroumov, *Inzh.-Fiz. Zh.*, **10**, No. 5, 683–695 (1966).
3. Yu. K. Stishkov and A. A. Ostapenko, *Élektron. Obrab. Mater.*, No. 4, 62–64 (1981).
4. Yu. K. Stishkov, *Élektron. Obrab. Mater.*, No. 6, 29–32 (1977).
5. J. Malcher, *Magn. Gidrodin.*, No. 2, 3–30 (1974).
6. A. I. Fedonenko and A. I. Zhakin, *Magn. Gidrodin.*, No. 3, 74–78 (1982).
7. A. I. Zhakin and I. E. Tarapov, *Magn. Gidrodin.*, No. 4, 53–57 (1979).
8. A. I. Zhakin, *Izv. Akad. Nauk SSSR*, *Mekh. Zhidk. Gaza*, No. 4, 137–142 (1979).
9. A. I. Zhakin and I. E. Tarapov, *Izv. Akad. Nauk SSSR*, *Mekh. Zhidk. Gaza*, No. 4, 20–26 (1981).
10. A. I. Zhakin, *Izv. Akad. Nauk SSSR*, *Mekh. Zhidk. Gaza*, No. 2, 14–20 (1988).
11. E. L. Tarunin and Yu. A. Yamshinina, *Magn. Gidrodin.*, No. 2, 142–144 (1990).
12. E. L. Tarunin and Yu. A. Yamshinina, *Izv. Ross. Akad. Nauk*, *Mekh. Zhidk. Gaza*, No. 3, 23–29 (1994).
13. A. A. Mikhailov and Yu. K. Stishkov, *Magn. Gidrodin.*, No. 2, 76–80 (1977).
14. E. L. Tarunin, *Computational Experiment in Free-Convection Problems* [in Russian], Irkutsk (1990).
15. E. L. Tarunin, *Izv. Akad. Nauk SSSR*, *Mekh. Zhidk. Gaza*, No. 5, 72–74 (1967).
16. D. L. Shvartsblat, *Izv. Akad. Nauk SSSR*, *Mekh. Zhidk. Gaza*, No. 5, 84–90 (1969).